From Topology to de Rham Cohomology: An Introduction to Smooth Manifolds and Differential Forms

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Contents

1 Topology and manifolds

1.1 What does topology study?

Topology is the study of properties of spaces that are preserved under continuous deformations, such as stretching, bending, and twisting, but not under tearing.

A standard illustrative example is that a coffee cup and a donut are considered topologically equivalent, whereas neither is equivalent to a pie plate.

Definition 1.1. Let X be a set. Then T is a *topology* on X if:

(i) $\emptyset, X \in T$;

(ii) $A, B \in T \implies A \cap B \in T$;

(iii) $A_i \in T$ for all $i \in I \implies \bigcup_{i \in I} A_i \in T$.

1.2 Open sets

Definition 1.2. Let $(X, \mathcal{T})$ be a topological space. $U \subset X$ is **open** if $U \in \mathcal{T}$

Examples in $\mathbb{R}$ (with the standard topology): $(0,1), (2,\infty), (1,2) \cup (5,9)$

Informally, a subset of $\mathbb{R}$ is open if every point contained in it admits a surrounding interval that lies entirely within the set.

For example, the sets $(0,1)$, $(2,\infty)$, and $(1,2) \cup (5,9)$ are all open in $\mathbb{R}$.

1.3 Continuity without epsilon-delta

Definition 1.3. A function $f : X \to Y$ between topological spaces is *continuous* if for every open set $V \subset Y$, the preimage

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

is open in X.

1.4 Homeomorphism

Definition 1.4. Let X and Y be topological spaces. A *homeomorphism* is a continuous bijection $f : X \to Y$ whose inverse $f^{-1} : Y \to X$ is also continuous. If such a map exists, X and Y are said to be *homeomorphic*, or topologically equivalent.

Informally, two spaces are homeomorphic if one can be continuously deformed into the other without tearing or cutting.

The following are standard illustrative examples:

- The circle and the square are homeomorphic.
- The circle and a line segment are not homeomorphic.

- The coffee mug and the torus (donut) are homeomorphic.

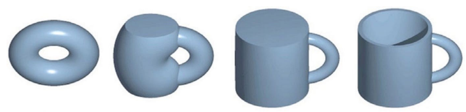

Figure 1: Homeomorphism from donut to coffee mug

1.5 Manifolds, informally

A **manifold** is a topological space that is *locally homeomorphic to* $\mathbb{R}^n$; that is, every point p admits an open neighbourhood that is homeomorphic to $\mathbb{R}^n$.

1.6 Curve vs surface

A *curve* is a one-dimensional manifold, locally homeomorphic to $\mathbb{R}^1$, and a *surface* is a two-dimensional manifold, locally homeomorphic to $\mathbb{R}^2$.

1.7 Disks: open vs closed as manifolds

Open disk: The open disk is locally Euclidean, being homeomorphic to $\mathbb{R}^2$. Since it has no boundary, every point admits a neighbourhood extending in all directions.

Closed disk: Every boundary point of the closed disk admits only a neighbourhood homeomorphic to a Euclidean half-plane, rather than to $\mathbb{R}^2$. Consequently, the closed disk is not a manifold in the strict sense (it is, however, an example of a *manifold with boundary*).

1.8 Coordinate charts and atlases

A **coordinate chart** is a map that assigns a region of a manifold to a corresponding region of Euclidean space. Coordinate charts are required to be:

- invertible (bijective);
- continuous;
- smooth (C^∞, i.e. infinitely differentiable).

An **atlas** is a collection of two or more overlapping charts that together cover the manifold.

Remark 1.5. A sphere requires at least two coordinate charts because it is a curved, closed, compact surface: no single flat 2D plane can be smoothly mapped onto the entire surface of a sphere without distortion or breakdown. ◇

1.9 Non-manifolds

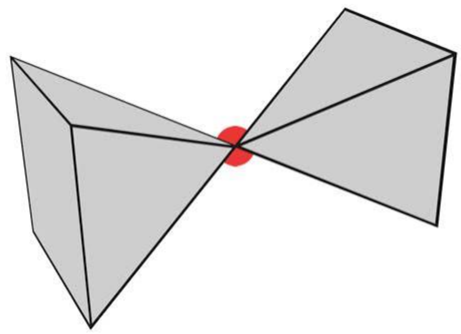

Figure 2: Example of a non-manifold

2 Smooth manifolds and tangent spaces

2.1 Continuous vs differentiable vs smooth

A function is **continuous** if the preimage of every open set under the function is open.

In general topology, there is no intrinsic notion of differentiability; the notion of a *smooth* structure, discussed below, must be introduced separately in order to speak of differentiable functions on a manifold.

2.2 Transition maps

If two coordinate charts overlap, the *transition map* between them is the function that converts coordinates expressed in one chart's system into coordinates expressed in the other's, on the region of overlap.

Definition 2.1. If chart φ maps a patch of M to $\mathbb{R}^n$ and chart ψ maps an overlapping patch to $\mathbb{R}^n$, the *transition map* is $\psi \circ \varphi^{-1}$, defined on the overlap.

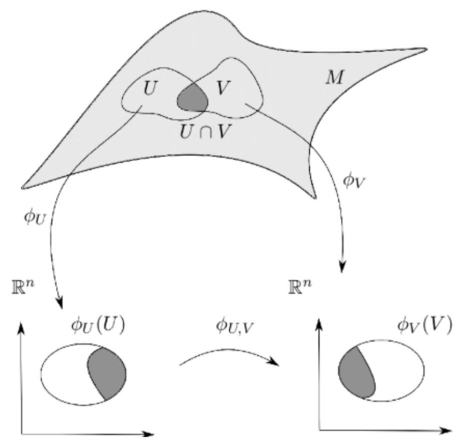

Figure 3: Visual representation of a transition map

2.3 Smooth manifolds

A manifold becomes a **smooth manifold** when all of its transition maps are smooth (infinitely differentiable).

Remark 2.2. A C^k **manifold** requires only that the transition maps be k-times continuously differentiable. ◇

2.4 Tangent vectors and tangent spaces

In $\mathbb{R}^n$, a **tangent vector** may be understood as the instantaneous direction of a curve at a given point.

On a smooth manifold, a **tangent vector** at a point p is defined analogously, as the instantaneous direction of a curve passing through p while remaining on the manifold.

The **tangent space** T_pM at a point p is the vector space consisting of all tangent vectors at p.

2.5 Smooth maps and diffeomorphisms

Definition 2.3 (Smooth map)**.** A map $f : M \to N$ is smooth if for every $p \in M$, expressing f in local coordinates gives an ordinary C^∞ map.

Definition 2.4 (Diffeomorphism)**.** A diffeomorphism $f : M \to N$ is a smooth map that is a bijection and whose inverse $f^{-1} : N \to M$ is also smooth.

Remark 2.5. Diffeomorphism $\implies$ homeomorphism, but homeomorphism $\not\Rightarrow$ diffeomorphism. ◇

3 Differential forms

3.1 Vector fields vs differential forms - the duality

A **vector field** assigns a tangent vector to each point of a manifold.

At each point p, the tangent vectors based at p form a vector space called the **tangent space**, T_pM.

A **differential form** is dual to a vector field. Whereas a vector field assigns a vector to each point, a differential form assigns to each point a linear functional that accepts a tangent vector as input and returns a real number.

At a point p, the space of all such linear functionals on T_pM is called the **cotangent space**, denoted T_p^*M.

3.2 The Differentials dx and dy as 1-Forms

In elementary calculus, dx and dy are typically interpreted as infinitesimal changes in x and y.

In differential geometry, dx is instead understood as a linear functional that takes a tangent vector as input and returns its x-component; it is an example of a 1-form.

3.3 Forms of different degrees on $\mathbb{R}^2$

A **0-form** is simply a smooth function, for example $f(x, y) = x^2 + y^2$.

A **1-form** assigns to each point a linear functional accepting one tangent vector, for example $\omega = x\,dy - y\,dx$.

A **2-form** assigns to each point a bilinear, alternating functional accepting two tangent vectors, for example $\eta = x\,dx \wedge dy$.

3.4 The wedge product

The **wedge product**, denoted $\wedge$, combines differential forms and is antisymmetric: $dx \wedge dy = -dy \wedge dx$.

It follows immediately that $dx \wedge dx = 0$.

3.5 The area form

The 2-form $dx \wedge dy$ is called the *area form* on $\mathbb{R}^2$, since evaluating it on a pair of tangent vectors returns the signed area of the parallelogram they span.

3.6 Pushforwards of tangent vectors

Given a smooth map $F : M \to N$, the **pushforward** F_* takes a tangent vector at a point of M and sends it to a tangent vector at the corresponding point of N. In other words, tangent vectors move with the map.

Tangent vectors push forward (same direction as the map).

Differential forms pull back (opposite direction to the map).

3.7 Pullbacks

Given a smooth map $F : M \to N$, the **pullback** F^* takes a differential form on the target manifold N and produces a differential form on the source manifold M. Intuitively, it tells us how a form on the target behaves when viewed through the map F. Unlike tangent vectors, differential forms move opposite to the direction of the map. This is the key distinction between pullbacks and pushforwards and is worth revisiting after Section 3.9.

3.8 Why differential forms matter for manifolds

Differential forms are defined using only tangent spaces, so they make sense on any smooth manifold without requiring an ambient $\mathbb{R}^n$, coordinates, or a metric. This makes them the natural objects for integration on manifolds. As discussed in Section 4, differential forms also lead to de Rham cohomology, which uses calculus to detect topological features such as holes in a space.

4 Exterior derivative, Stokes, de Rham

4.1 The exterior derivative

The **exterior derivative**, denoted d, generalizes differentiation to differential forms. It increases the degree of a form by one:

$$\text{0-form} \to \text{1-form} \to \text{2-form} \to \cdots$$

4.2 Ordinary derivative vs exterior derivative

The ordinary derivative acts only on functions, whereas the exterior derivative acts on differential forms of every degree. It provides a single operation that unifies the concepts of gradient, curl, and divergence from vector calculus.

4.3 Closed and exact forms

A differential form α is

- **closed** if $d\alpha = 0$;
- **exact** if $\alpha = d\beta$ for some form β of one lower degree.

4.4 Exact $\implies$ closed

If

$$\alpha = d\beta, \tag{4.1}$$

then

$$d\alpha = d(d\beta) = 0, \tag{4.2}$$

since the exterior derivative always satisfies

$$d^2 = 0. \tag{4.3}$$

4.5 Is every closed form exact?

Not necessarily. Whether every closed form is exact depends on the topology of the underlying space; this question is precisely what de Rham cohomology, introduced below, is designed to measure.

4.6 Stokes' theorem, informally

Stokes' theorem states that

$$\int_M d\omega = \int_{\partial M} \omega. \tag{4.4}$$

In words, integrating the derivative of a form over a region is equivalent to integrating the form itself over the boundary. It is the higher-dimensional analogue of the Fundamental Theorem of Calculus.

4.7 Relation to the Fundamental Theorem of Calculus, Green's Theorem, and the Divergence Theorem

Many familiar theorems from calculus are special cases of Stokes' theorem, as summarized in Table 1.

Classical theorem	**Special case of Stokes' theorem on...**
Fundamental Theorem of Calculus	An interval in $\mathbb{R}$
Green's theorem	A region in $\mathbb{R}^2$
Divergence theorem	A region in $\mathbb{R}^3$

Table 1: Classical integral theorems as special cases of Stokes' theorem.

4.8 de Rham cohomology, informally

The k-th de Rham cohomology group is

$$H^k = \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}. \tag{4.5}$$

If every closed form is exact, then $H^k = 0$. Otherwise, nonzero cohomology indicates the presence of a k-dimensional hole.

4.9 Why cohomology detects holes

Closedness is a local condition because it depends only on derivatives, while exactness is a global condition requiring a single potential function over the entire space. Holes create global obstructions that prevent closed forms from being exact.

4.10 H^0 and H^1

H^0 measures the number of connected components of a space.

H^1 measures one-dimensional holes, corresponding to loops that cannot be continuously filled in.

4.11 The circle S^1 and its cohomology

The circle contains one one-dimensional hole, so

$$H^1(S^1) \cong \mathbb{R}, \tag{4.6}$$

which is nontrivial. This reflects the fact that a loop around the circle cannot be shrunk to a point.

Remove later

Closed = Local condition

Exact = Global condition

H^0: number of connected components H^1: 1D holes/loops that can't be filled in

$H^1(S^1) \cong \mathbb{R}$- nontrivial